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A systematic procedure for constructing critical state models in three dimensions

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Abstract

A general procedure for developing constitutive models for frictional materials possessing a critical state is developed in a three-dimensional context. The procedure starts from the laws of thermo-dynamics, so that the first and second laws of thermo-dynamics are automatically satisfied. There is hence no need to invoke any extraneous stability postulates. The models involve a number of parameters, which can be interpreted in terms of micro-mechanical energy storage and dissipative mechanisms. In most cases non-associated flow rules are predicted and in some cases the yield surfaces are seen to have concave segments. The procedure is more general than that traditionally used for materials with non-associated flow rules, in that plastic potentials are not needed and not presumed to exist. In illustration, examples of families of models are given in which the critical state surface is either the Drucker–Prager or the Matsuoka–Nakai cone.

Keywords: Elastic/plastic; Thermo-mechanics; Hyper-plasticity; Non-associated flow rule; Critical state

1. Introduction

The critical state models developed by Schofield and Wroth (1968) and Roscoe and Burland (1968) and co-workers are one of the cornerstones of modern geomechanics. The early theories have been developed in a number of directions. In particular Nakai and Mihara (1984), Salgado and Byrne (1991) and Matsuoka et al. (1999) have developed three-dimensional critical state models using the spatially mobilized plane concept of Matsuoka and Nakai (1974)—see also Matsuoka (1976). The purpose of the current paper is to describe a procedure for constructing such three-dimensional critical state models, which is more general than those proposed hitherto, although the methods employed have some formal similarities with those used by Matsuoka et al. (1999). The procedures of the present paper are physically based and it is possible to assign physical meanings to the various parameters and to the steps in the formal procedure.

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The standard procedure for constructing elastic/plastic models for soils, clays, sands and other geomaterials is to specify the elasticity law, yield condition, plastic potential (which is needed since in most cases non-associated flow rules are required to fit observational data), and the isotropic and kinematic hardening laws all *independently of each other*. In most cases there is no discussion of restrictions posed by the laws of thermo-dynamics, although frequently reference is made to various "stability postulates". Even though most geotechnical problems are effectively isothermal and temperature changes are small, any sort of failure, of necessity, involves large changes in entropy, so that thermo-dynamic ideas cannot really be ignored. In the last twenty years, there have been many important advances in our understanding of thermo-mechanics, as described in the texts by Ziegler (1983) Lemaitre and Chaboche (1990) and Maugin (1992, 1999) and the papers by Germain et al. (1983), Ziegler and Wehrli (1987) Reddy and Martin (1994) and Houlsby and Puzrin (2000). Whilst these ideas have been widely applied to problems in the mechanics of metals, it was frequently stated that these theories could not be applied to geomaterials, because they could not embrace materials with non-associated flow rules. However Collins and Houlsby (1997) showed that such flow rules appear naturally from the thermo-mechanical theory, when the material deformations are "frictional", in the sense that the material response is governed by stress ratios, rather than by the magnitude of the stresses themselves. Le Pape and Sieffert (2001) have recently used these ideas to model the interaction between shallow foundations and frictional soils.

A major attraction of the thermo-mechanical approach is that it provides a tight mathematical structure and removes much of the arbitrariness of the conventional approach. The whole of the constitutive structure (yield condition, flow rule, hardening laws and elasticity law) is determined from just two thermo-dynamic potentials. This procedure has recently been used by Collins and Kelly (2002), Collins and Hilder (2002) and Collins and Muhunthan (2003) to develop a number of new classes of critical state constitutive models as well as to produce a number of new general theoretical results, in the context of tri-axial tests. Some of these ideas are generalized in this paper to general three-dimensional situations.

2. The structure of the thermo-mechanical theory

A good starting point for analyzing *isothermal* deformations of geomaterials is the statements of the first and second laws of thermo-dynamics in the form

$$\widehat{W} = \dot{\Psi} + \Phi, \quad \text{where } \Phi \geqslant 0 \tag{1}$$

where $\widehat{W} \equiv \mathrm{tr}(\sigma \dot{e}) \equiv \sigma : \dot{e}$ is the rate of working, per unit volume, of the effective stress in a continuum element. (The ":"product notation stands for the double inner product, so that $\sigma : \dot{e} \equiv \sigma_{ij}\dot{e}_{ij}$.) In this paper all stresses are understood to be effective stresses, so we shall not use the "prime" notation. Compressive stresses and strains will be taken to be positive. The function Ψ is the (Helmholtz) free energy, and Φ is the rate of dissipation; both defined per unit volume. The free energy Ψ is a function of the state variables, which we will take as the elastic e^{e} and plastic e^{p} strains, but Φ and $\widehat{W}$ are not the time derivatives of state functions. This choice of state variables is sufficient to describe the volumetric and shear hardening associated with critical state models, but extra internal state variables, describing the history of the deformation are needed to describe more complex hardening behaviors, particularly those associated with cyclic loading. For rate independent, elastic/plastic materials, Φ, as well as depending on the state variables, is also a homogeneous function of degree 1 in the plastic strain-rate tensor $\dot{e}^{\mathrm{p}}$. The inequality in (1) must be "strictly greater than" whenever irreversible, plastic deformations are occurring.

In general, the free energy will depend on both the elastic and plastic strains. An important special case is where the free energy can be expressed as the sum of a function of only elastic strains, plus a function of only plastic strains:

$$\Psi = \Psi_1(e^{\mathrm{e}}) + \Psi_2(e^{\mathrm{p}}) \tag{2}$$

This is a necessary and sufficient condition for the model to be "decoupled" in the sense that the resulting instantaneous elastic modulus tensor is independent of the plastic strain. Aspects of the analogous theory for coupled materials have been developed by Maier and Hueckel (1977), Valanis and Peters (1993), Collins and Houlsby (1997) and Collins (2002).

The total rate of working associated with the effective stress can be written as the sum of an elastic and plastic component, which using (1) and (2) can be written

$$\widehat{W}^{\mathrm{e}} \equiv \sigma : \dot{e}^{\mathrm{e}} = \dot{\Psi}_1 = \frac{\partial \Psi_1(e^{\mathrm{e}})}{\partial e^{\mathrm{e}}} : \dot{e}^{\mathrm{e}} \tag{3}$$

and

$$\widehat{W}^{\mathrm{p}} = \sigma : \dot{e}^{\mathrm{p}} = \dot{\Psi}_2 + \Phi = \frac{\partial \Psi_2(e^{\mathrm{p}})}{\partial e^{\mathrm{p}}} : \dot{e}^{\mathrm{p}} + \frac{\partial \Phi}{\partial \dot{e}^{\mathrm{p}}} : \dot{e}^{\mathrm{p}} \tag{4}$$

where the last term in (4) follows from Euler's Theorem for homogeneous functions of degree 1. From (3) we can deduce the basic elastic part of the constitutive law:

$$\sigma = \frac{\partial \Psi_1(e^{\mathrm{e}})}{\partial e^{\mathrm{e}}} \tag{5}$$

whilst (4) is satisfied if

$$\sigma = s + \chi, \quad \text{where } s \equiv \frac{\partial \Psi_2(e^{\mathrm{p}})}{\partial e^{\mathrm{p}}} \quad \text{and} \quad \chi \equiv \frac{\partial \Phi}{\partial \dot{e}^{\mathrm{p}}} \tag{6}$$

The stress tensors s and χ are termed the *shift* (*drag*, *back or quasi-conservative*) *stress* and *dissipative stress* respectively. Eq. (6) cannot be deduced formally from (4), in the same way that (5) follows from (3), because the dissipative stress χ depends on the plastic strain rate $\dot{e}^{\mathrm{p}}$. The validity of (6) has the status of a constitutive postulate, it is an example of "Ziegler's orthogonality postulate". It is equivalent to the maximum entropy production criterion, widely invoked in irreversible thermo-dynamics, and is necessary to obtain a unique formulation. It is a very weak assumption and the resulting class of material models includes all the major continuum models of thermo-mechanics. Eq. (4) for the plastic work rate can hence also be rewritten:

$$\widehat{W}^{\mathrm{p}} \equiv \sigma : \dot{e}^{\mathrm{p}} = \dot{\Psi}_2 + \Phi = s : \dot{e}^{\mathrm{p}} + \chi : \dot{e}^{\mathrm{p}} \tag{4'}$$

Thus whilst the plastic work is the product of the *true stress* with the plastic strain rate, the dissipation rate is the product of the *dissipative stress* with the plastic strain rate. These are only equal if the shift stress is zero, or, equivalently, if the free energy depends only on the elastic strains. The decomposition of the total stress in (6) is most familiar in kinematic hardening models for *anisotropic* materials, where the shift or back stress defines the moving "center" of the yield locus. However, at least within the confines of single surface, critical state models, it is necessary to introduce such a shift stress to describe the *isotropic* compression and expansion of a material with different yield stresses in isotropic compression and expansion (decompression). This point has been elaborated upon by Collins and Kelly (2002) and Collins and Hilder (2002).

The plastic work associated with Ψ_2 is hence *stored and not dissipated*. This stored plastic work arises out of the inhomogeneous nature of the deformation on the *micro-scale*, as discussed in a general context by Mroz (1963, 1973) and Besseling and van der Giessen (1993). Although from a continuum viewpoint, all points in a given element are at yield and deforming plastically, viewed on a micro-scale only part of this element is undergoing plastic deformations, the remainder is still below yield and responding elastically. The *plastic strains are not kinematically compatible*, so that upon unloading some *residual elastic strains* are still present, so that the *total residual strain*, comprising the plastic and residual elastic strains, is kinematically compatible. See Mroz (1973) or Maugin (1992) for a full discussion of this point. Some of the

energy associated with these micro-level elastic deformations is stored and not dissipated, but this energy is "locked in" and can only be recovered when the whole continuum element is subjected to reversed plastic loading at the macro-continuum level.

Collins and Kelly (2002) and Collins and Hilder (2002) suggested that in the case of granular materials the micro-level plastic energy dissipation *during normal compaction* arises from the plastic deformations occurring at the inter-granular contacts occurring on the "strong" force chains which are bearing the bulk of the applied loads, and that the "locked in" elastic energy is produced in the "weak" sub-networks, where the local stresses are not large enough to produce plastic deformations at the grain contacts. In contrast during *distortional deformations*, most of the energy dissipation occurs in the "weak" sub-networks, since shear deformations are much easier to induce there, than in the force chains. Very little, if any, of the distortional plastic work is stored. This view of the micro-structure of granular media is based on that described by Radjai et al. (1997). More generally the plastic strains can be associated with the irreversible rearrangement of the particles, whilst the elastic energy arises from the elastic compression of the particle contacts. Some of this elastic energy will be released during the swelling which occurs upon unloading, but part of this energy will be "trapped" as a result of the irreversible changes in the particles configuration.

Very few authors seem to have discussed the possible role of stored energy in geomechanics. The exceptions would appear to be Palmer (1967) who discussed this issue in relation to the early Cam-clay models, Houlsby (1981) who pioneered the application of modern thermo-mechanics to geomechanics, Collins and Houlsby (1997) and Jefferies (1997), who noted that data for dense Erksak sand indicated that about one quarter of the applied plastic work was not being dissipated. Although this issue has not been much discussed explicitly, most of the extant models do actually involve both stored and dissipated plastic work. For example, under normal compression of modified Cam clay, one half of the plastic work is stored and one half is dissipated. Collins and Muhunthan (2003) have recently demonstrated that, provided anisotropy is included in the model, dilatant "quasi-steady" shear deformations occur *without* any plastic work being stored, but that the shift stress components are nevertheless non-zero.

3. The development of the elastic/plastic constitutive laws

One of the main attractions of the thermo-mechanical approach is that the yield condition and flow rule can be simply deduced from the specified dissipation function Φ. However since this function is the product of the plastic strain rate with the dissipative stress χ and not the true stress σ, as proved in (4′), this yield function and flow rule are first formulated in dissipative stress space and not true stress space. Formally this yield function is the Legendre dual of the dissipation function. In the examples discussed here, however, the derivation of the yield condition and flow rule is straight forward and does not need the explicit use of Legendre transforms.

3.1. Standard and non-standard materials

In the present context a "*standard material*" is defined to be one in which the dissipation function is of the form $\Phi(e^{\mathrm{p}}, \dot{e}^{\mathrm{p}})$ and is independent of the stress and elastic strain. The general theory then demonstrates that the associated dissipative stress χ defined in (6) must satisfy a convex yield condition of the form $f(e^{\mathrm{p}}, \chi) = 0$, whenever the dissipation is non-zero, and that the flow rule is *associated* in the sense that the plastic strain rate is normal to this yield condition in dissipative stress space. The yield condition in true stress space is then found simply by putting $\chi = \sigma - s$, where s is deduced from the free energy function as in (6). This transformation is a simple *translation* and the associativity of the flow rule is preserved. Such models are widely used to describe the behavior of metals.

For *frictional materials* however, the dissipation function must also depend parametrically on the true stress tensor. This is because the plastic behavior is governed by critical values of the *ratio* of stress components (e.g. friction angles) and not by the *magnitude* of individual stress components (e.g. yield shear stresses). The dissipation function, however, has the dimensions of stress/time and hence must contain a variable with the dimensions of stress, in addition to the plastic strain and strain rate. Since no material parameter is available, it is necessary to include one or more true stress variables, such as the mean pressure. Thus in general for such "*non-standard materials*" the dissipation function is of the form $\Phi(\sigma, e^{\mathrm{p}}; \dot{e}^{\mathrm{p}})$, or equivalently $\Phi(e^{\mathrm{e}}, e^{\mathrm{p}}; \dot{e}^{\mathrm{p}})$, where the elasticity law (5) has been used to replace σ by e^{e}. The resulting yield condition in dissipative stress space is hence of the form $f(\sigma, e^{\mathrm{p}}, \chi) = 0$, and has a parametric dependence on σ. As a consequence the transformation from dissipative stress space to true stress space is no longer a pure translation, and *the yield surface distorts* when transformed into true stress space and *the flow rule is no longer associated*. In this sense such frictional materials are "*non-standard*", but nevertheless the shape of the yield surface and the flow rule can still be determined uniquely.

Elastic/plastic materials, whose properties can be derived from thermo-mechanical potentials have been termed *hyper-plastic*, by Houlsby and Puzrin (2000), by analogy with the corresponding concept in elasticity theory. We note from (6) that for an isotropic material, the shift stress tensor is coaxial with the plastic strain tensor. If the dissipation function does not depend on stress as in the "standard" non-frictional model, then the dissipative stress is coaxial with the plastic strain-rate tensor. However if Φ depends on the true stress σ as well as on $\dot{e}^{\mathrm{p}}$, this is no longer necessarily true, since Φ could depend on the joint invariants of these two tensors. This point is discussed further below.

4. Development of critical state models

A comprehensive family of models for frictional materials can be generated by starting with a dissipation potential in the form:

$$\Phi = \sqrt{(A\dot{e}_{\mathrm{v}}^{\mathrm{p}})^2 + (B\dot{\bar{e}}_{\gamma}^{\mathrm{p}})^2} \tag{7}$$

where $e_{\mathrm{v}}^{\mathrm{p}} \equiv \mathrm{tr}(e^{\mathrm{p}})$ is the volumetric plastic strain, and $\bar{e}_{\gamma}^{\mathrm{p}}$ is a suitably chosen plastic shear strain—assumed non-negative, by convention. In plane strain or axial symmetry, this strain would be proportional to the difference between the two distinct, principal strain components. In three dimensions one could choose $\bar{e}_{\gamma}^{\mathrm{p}}$ to be $\|\mathrm{dev}\, e^{\mathrm{p}}\|$, the magnitude of the deviatoric plastic strain, which is $\sqrt{3}$ times the shear strain on the octahedral plane. Alternatively, as will be seen below, it could be the shear strain on some other "failure plane". In using such a dissipation function we are recognizing that energy dissipation results from two fundamental mechanisms, namely isotropic compaction or dilation, and frictional shearing.

The coefficient functions A and B are homogeneous functions of degree one in the true stress. For isotropic, cohesionless materials it is sufficient to assume that A and B depend on the mean effective pressure p and also on the consolidation pressure p_{c}. The consolidation pressure is taken to be a known function of the volumetric strain as in the classical theory. The specific form of these functions does not need to be specified at this stage. The dissipative stress tensor is now found by differentiating Φ with respect to the plastic strain rate, and using (6). The dissipative stress invariants, *conjugate* to the volumetric and shear strains are hence:

$$\pi = \partial\Phi/\partial\dot{e}_{\mathrm{v}}^{\mathrm{p}} = A^2\dot{e}_{\mathrm{v}}^{\mathrm{p}}/\Phi \quad \text{and} \quad \bar{\tau} = \partial\Phi/\partial\dot{\bar{e}}_{\gamma}^{\mathrm{p}} = B^2\dot{\bar{e}}_{\gamma}^{\mathrm{p}}/\Phi \tag{8}$$

so that:

$$\Phi = \pi\dot{e}_{\mathrm{v}}^{\mathrm{p}} + \bar{\tau}\dot{\bar{e}}_{\gamma}^{\mathrm{p}} \tag{9}$$

in agreement with (4′). It follows further that $\pi = \frac{1}{3}\mathrm{tr}\,\chi$ is the "dissipative pressure". The precise significance of $\bar{\tau}$ depends on the definition of the plastic shear strain $\bar{e}^{\mathrm{p}}_{\gamma}$, and is hence model dependent.

The form of the yield condition in dissipative $(\pi, \bar{\tau})$ stress space, follows by eliminating the strain rates between (7) and (8):

$$\frac{\pi^2}{A^2} + \frac{\bar{\tau}^2}{B^2} = 1 \tag{10}$$

The yield loci in dissipative stress space are hence concentric ellipses, centered on the origin and with semi-axes of length A and B—see Fig. 1. This result is deceptively simple. As will be seen, it is still possible to generate many families of models in two and three dimensions, by choosing different forms of the functions A and B, different forms for the shift stress (or free energy function) and making different choices for the plastic shear strain.

The flow rule in dissipative stress space is found by eliminating Φ between the two parts of (8):

$$\tan\psi = \frac{-\dot{e}^{\mathrm{p}}_{\mathrm{v}}}{\dot{\bar{e}}^{\mathrm{p}}_{\gamma}} = -\frac{\pi}{\bar{\tau}}\frac{B^2}{A^2} \tag{11}$$

where ψ is a dilation angle. As is readily verified, the direction of the plastic strain-rate vector is normal to the yield ellipse in dissipative stress space, as required by the general theory.

As in conventional critical state theory, we will assume that the material exhibits volumetric hardening so that the plastic free energy function depends on $e^{\mathrm{p}}_{\mathrm{v}}$, but not on $\bar{e}^{\mathrm{p}}_{\gamma}$. (Collins and Hilder (2002) have shown that when shear strain hardening is included, the yield surfaces must rotate and the material response is anisotropic.) As a result of this assumption, the shift stress tensor must be isotropic with a shift pressure $\rho \equiv \frac{1}{3}\mathrm{tr}\,s$, where $\rho = \rho(e^{\mathrm{p}}_{\mathrm{v}}) = \mathrm{d}\Psi_2(e^{\mathrm{p}}_{\mathrm{v}})/\mathrm{d}e^{\mathrm{p}}_{\mathrm{v}}$, whilst the deviatoric part of the shift stress is zero. Hence

$$p = \rho + \pi \quad \text{and} \quad \bar{q} = \bar{\tau} \tag{12}$$

where $\bar{q}$ is the, model dependent, true shear stress, conjugate to $\bar{e}^{\mathrm{p}}_{\gamma}$.

4.1. Conditions on the normal consolidation line

On the normal consolidation line $\dot{\bar{e}}^{\mathrm{p}}_{\gamma}$ is zero, so using (8), the yield condition (10) and flow rule (11), we deduce that:

$$\bar{\tau} = 0, \quad \bar{q} = 0, \quad \pi = \pm A, \quad \text{and} \quad p = \rho \pm A \tag{13}$$

The function A is hence determined by the yielding behavior of the material under isotropic compression and expansion. The two values for p given by the last equation (13) are the yield pressures under normal

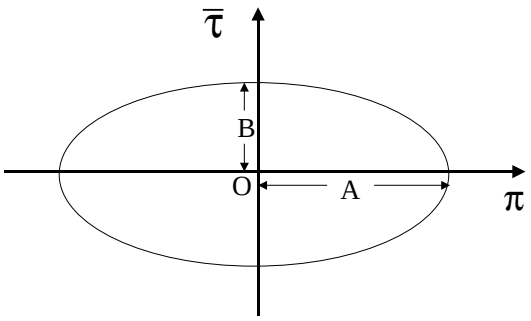

Fig. 1. Elliptical yield locus in dissipative stress space.

compression and expansion respectively. If we denote the consolidation pressure by p_c, and assume, as usual for cohesionless soils, that yielding under expansion occurs at zero effective pressure, we deduce that:

$$\rho = A|_{p=0} \quad \text{and} \quad p_c = \rho + A|_{p=p_c} \tag{14}$$

The function A is homogeneous of degree one in the two pressure variables p and p_c. If we further assume, for simplicity, that A is in fact *linear* in these two variables, it follows from (14), that:

$$\rho = \tfrac{1}{2}\gamma p_c \quad \text{and} \quad A = (1-\gamma)p + \tfrac{1}{2}\gamma p_c \tag{15}$$

where γ is a parameter, whose value lies in $[0, 1]$. For modified Cam clay $\gamma = 1$, and A depends only on the consolidation pressure. See Collins and Hilder (2002) for a full discussion of this model of isotropic deformations.

4.2. Conditions on the critical state surface (cone)

Dually on the critical state surface, the volumetric plastic strain rate $\dot{e}_v^p$ is zero, so that Eqs. (8), (10) and (11) now imply:

$$\pi = 0, \quad p = \rho, \quad \text{and} \quad \bar{\tau} = \bar{q} = B \tag{16}$$

where we have rejected the "$-B$" solution, since, by convention, we will always define the shear stress invariants to be non-negative. The last equation in (16) shows that the critical state surface is given by $\bar{q} = B$. The function B hence determines the shape of this cone. It is to be emphasized that since we have yet to specify the shear strain $\bar{e}_\gamma^p$, $\bar{q}$ too, has yet to be specified. Only in the special case of the Drucker–Prager model, is $\bar{q}$ independent of the Lode angle and equal to $q \equiv \|\mathrm{dev}\,\sigma\|$, the standard deviatoric stress invariant. The cross-section of the critical state surface is only circular in this special case. From (15) and (16) it follows that the yield surface cuts the critical state surface on a plane, Π_c say, which is normal to the normal consolidation line (i.e. the principal diagonal in principal stress space) and on which the pressure is:

$$p_{css} = \rho = \tfrac{1}{2}\gamma p_c \tag{17}$$

The critical state pressure is hence equal to the shift pressure and is a fraction $\frac{1}{2}\gamma$ of the consolidation pressure. The ratio $p_c/p_{css} = r$, say, is sometimes referred to as the "spacing ratio" as in Yu (1998). Yu would appear to be the first author to recognize the importance of the spacing ratio as a constitutive parameter, and his CASM model has much in common with the present theory. The main difference is that his flow rules are based upon Rowe's stress–dilatancy relation, where as in the thermo-mechanical models, the flow rules are defined from a dissipation function. When the flow rule is determined from an extraneous postulate, such as a stress–dilatancy relation, it is not easy to determine the corresponding dissipation function and hence the associated micro-mechanical energy dissipation mechanisms.

For these models $\gamma = 2/r$. Since the shift stress defines the center of a kinematically hardening model, the center of these generalized critical state models is the point where the plane Π_c intersects the principal diagonal as illustrated in Fig. 2. During isotropic compression the part of the plastic work rate $p_c\dot{e}_v^p$, which is being stored is $\rho\dot{e}_v^p = \frac{1}{2}\gamma p_c\dot{e}_v^p = \frac{1}{r}p_c\dot{e}_v^p$, so that a fraction $1/r$ of the compressive work rate, is being stored. For modified Cam clay $r = 2$ and a half of the applied plastic work is being stored. This fraction is appreciably less than one half for sands and granular materials, which have a significantly higher spacing ratio. Jefferies' (1997) data would suggest a value of at least 4. The experimentally determined yield loci of Lee and Coop (1997) for decomposed granite soil and McDowell et al. (2002) for silica sand, have spacing ratios r of between 4 and 5. We can thus conclude that the yield surface becomes more elongated, in the sense that r increases, as the proportion of the compressive plastic work, which is being dissipated increases. There is no *stored plastic shear work* in a volumetric hardening model, since the free energy function does not depend on the plastic shear strain.

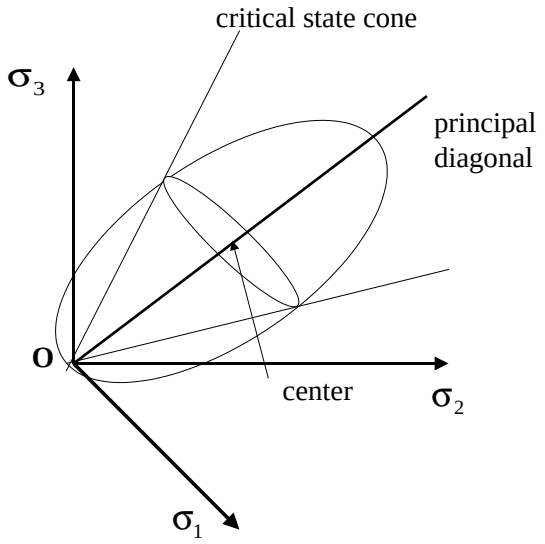

Fig. 2. Shows the "centre" of shifted yield surface in principal stress space.

By analogy with (15) the function B can also be written as a linear function of p and p_c:

$$B = \mu\left[(1-\alpha)p + \tfrac{1}{2}\alpha\gamma p_c\right] \equiv \mu[(1-\alpha)p + \alpha p_{css}] \tag{18}$$

where α, is the dimensionless parameter in the interval $[0,1]$ introduced by Collins and Kelly (2002). The parameter μ is also dimensionless and plays the role of a friction coefficient. On the critical state surface $p = p_{css} = \frac{1}{2}\gamma p_c$, so that $B = \mu p$, so that from (16) the equation of this surface is simply:

$$\bar{q} = \mu p \tag{19}$$

which is the equation of a *cone* in principal stress space, with the principal diagonal as axis.

4.3. Yield condition and flow rule in true stress space

The yield stress and flow rule in true stress space can now be found by rewriting (7) in terms of true stresses using Eqs. (12), (15) and (18)

$$\frac{\left(p - \frac{1}{2}\gamma p_c\right)^2}{\left[(1-\gamma)p + \frac{1}{2}\gamma p_c\right]^2} + \frac{\bar{q}^2}{\mu^2\left[(1-\alpha)p + \frac{1}{2}\alpha\gamma p_c\right]^2} = 1 \tag{20}$$

and

$$\tan\psi \equiv -\frac{\dot{e}_v^p}{\dot{\bar{e}}_\gamma^p} = -\mu^2\,\frac{\left(p - \frac{1}{2}\gamma p_c'\right)}{\bar{q}}\,\frac{\left[(1-\alpha)p + \frac{1}{2}\alpha\gamma p_c\right]^2}{\left[(1-\gamma)p + \frac{1}{2}\gamma p_c\right]^2} \tag{21}$$

Although these equations are complex, the yield surfaces in $(p,\bar{q})$ space are readily constructed using the parametric equation for an ellipse, since from (20)

$$p = \tfrac{1}{2}\gamma p_c + \left[(1-\gamma)p + \tfrac{1}{2}\gamma p_c\right]\cos\omega \quad \text{and} \quad \bar{q} = \mu\left[(1-\alpha)p + \tfrac{1}{2}\gamma\alpha p_c\right]\sin\omega \tag{22}$$

where ω is the standard parametric angle for an ellipse. These equations can be solved for the true stress variables to give:

$$p = \frac{1}{2}\gamma p_c \frac{(1+\cos\omega)}{(1-(1-\gamma)\cos\omega)} \quad \text{and} \quad \bar{q} = \frac{1}{2}\gamma p_c \mu \sin\omega \frac{(1+(1-2\alpha+\alpha\gamma)\cos\omega)}{(1-(1-\gamma)\cos\omega)} \tag{23}$$

These equations are formally identical with those derived by Collins and Hilder (2002) for tri-axial test models. The difference is in the meaning of the shear invariants $\bar{e}^{\mathrm{p}}_{\gamma}$ and $\bar{q}$.

The parameter $\alpha = 1$, in the case of modified Cam clay. As was shown in Collins and Kelly (2002) and Collins and Hilder (2002), the effect of decreasing the value of α is to make the yield surfaces (loci) more "tear-drop" shaped, and increase the degree of non-associativity in the flow rule. For very small values of this parameter (<0.172) the yield surfaces are *concave* near the origin, and dilatant deformations can occur *inside* the critical state surface. In the limiting case of $\alpha = 0$, *all* the yield surfaces lie *entirely* inside the critical state surface. Such models predict dilation during the hardening phase of a drained test. Such behavior is observed for medium to dense sands as discussed by Asaoka et al. (2001), who note that such behavior is "out of the scope of Cam-clay models". Whilst this is true, it is certainly not out of the scope of

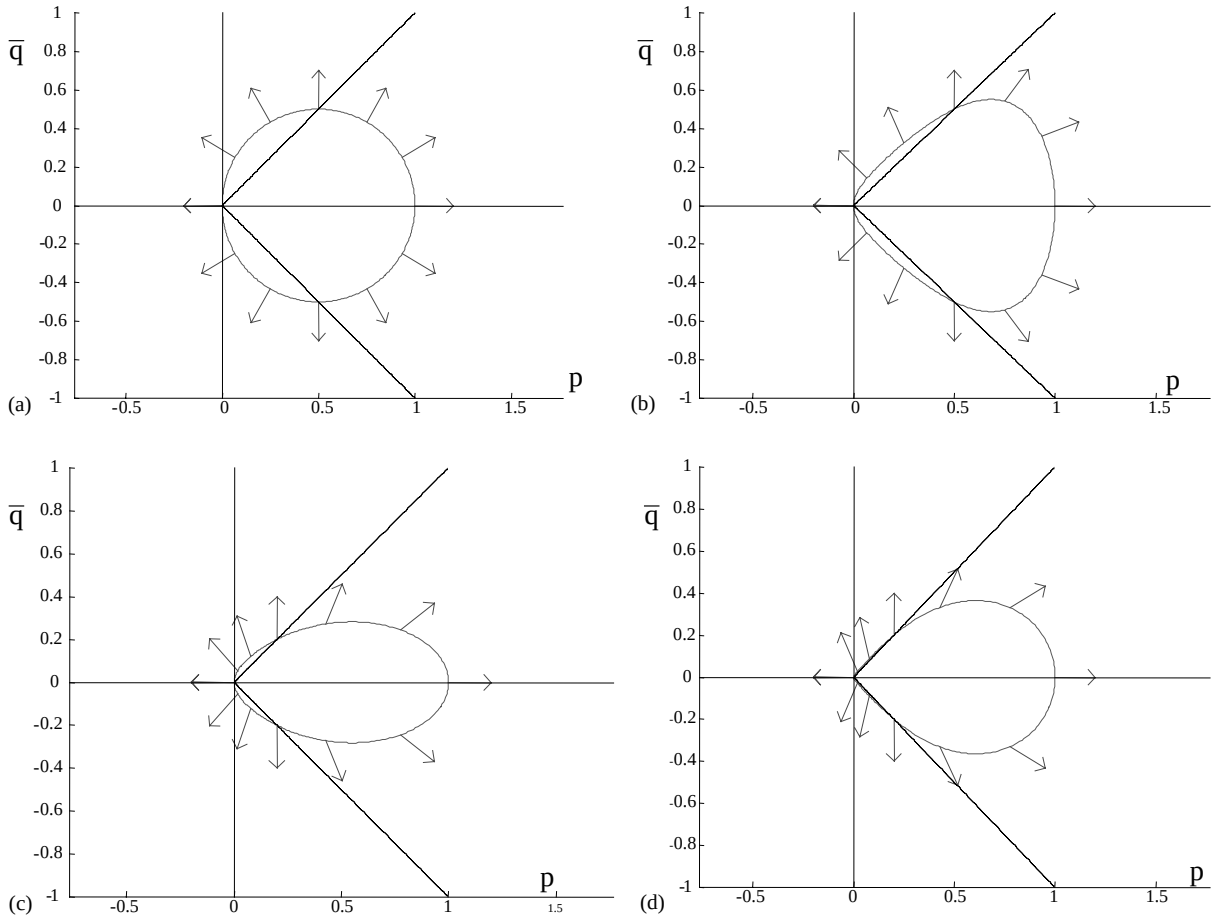

Fig. 3. Yield loci in $(p, \bar{q})$ space, for (a) $r = 2$, $\alpha = 1$ (Modified Cam clay); (b) $r = 2$, $\alpha = 0.5$; (c) $r = 5$, $\alpha = 0.5$; (d) $r = 5$, $\alpha = 0.2$, and $M = 1$.

this, more general, family of critical state models. Decreasing the value of the other model parameter γ, increases the spacing ratio and the yield loci become more elongated. Some representative yield loci in $(p, \bar{q})$ space, for different combinations of α and γ, are shown in Fig. 3, where the stresses are all non-dimensionalized by the consolidation pressure.

5. The Drucker–Prager model

The Drucker–Prager model is based on notions of failure and shearing on the octahedral plane. We first recall some elementary properties of the tractions and strain-rate components on this plane. The magnitudes of the normal and tangential plastic strain-rate components on this plane are:

$$\dot{e}^{\mathrm{p}}_{\mathrm{n,oct}} = \tfrac{1}{3}\mathrm{tr}\,\dot{e}^{\mathrm{p}} = \tfrac{1}{3}\dot{e}^{\mathrm{p}}_{\mathrm{v}} \quad \text{and} \quad \dot{e}^{\mathrm{p}}_{\gamma,\mathrm{oct}} = \tfrac{1}{\sqrt{3}}\|\mathrm{dev}(\dot{e}^{\mathrm{p}})\| \equiv \frac{1}{\sqrt{3}}\sqrt{\mathrm{tr}[(\mathrm{dev}\,\dot{e}^{\mathrm{p}})^2]} \tag{24}$$

Similarly the normal traction component on this plane is equal to the mean pressure, and the shear traction component is $1/\sqrt{3}$ times the magnitude of the deviatoric stress.

5.1. Development of yield surfaces and critical state cone

We will define the plastic shear strain rate used in Eq. (7) for the dissipation function to be just the magnitude of the deviatoric plastic strain rate, so

$$\dot{\bar{e}}^{\mathrm{p}}_{\gamma} \equiv \|\mathrm{dev}\,\dot{e}^{\mathrm{p}}\| = \sqrt{3}(\dot{e}^{\mathrm{p}}_{\mathrm{oct}}) \tag{25}$$

and hence

$$\Phi = \sqrt{A^2(\mathrm{tr}\,\dot{e}^{\mathrm{p}})^2 + B^2\|(\mathrm{dev}\,\dot{e}^{\mathrm{p}})\|^2} \equiv \sqrt{A^2(\mathrm{tr}\,\dot{e}^{\mathrm{p}})^2 + B^2\mathrm{tr}[(\mathrm{dev}\,\dot{e}^{\mathrm{p}})^2]} \tag{7'}$$

Using (6) the dissipative stress tensor is obtained by differentiating the dissipation function in (7′) with respect to the plastic strain rate, giving:

$$\chi = \{A^2(\mathrm{tr}\,\dot{e}^{\mathrm{p}})I + B^2(\mathrm{dev}\,\dot{e}^{\mathrm{p}})\}/\Phi \tag{26}$$

The mean pressure, deviatoric stress, and shear stress invariant of the *dissipative stress tensor* are hence given by:

$$\pi \equiv \tfrac{1}{3}\mathrm{tr}\,\chi = A^2(\mathrm{tr}\,\dot{e}^{\mathrm{p}})/\Phi, \quad \mathrm{dev}\,\chi = B^2(\mathrm{dev}\,\dot{e}^{\mathrm{p}})/\Phi \quad \text{and} \quad \bar{\tau} = \tau \equiv \|\mathrm{dev}\,\chi\| = B^2\dot{\bar{e}}^{\mathrm{p}}_{\gamma}/\Phi \tag{27}$$

which conforms with the general formulation in (8). In this model τ is the magnitude of the deviatoric dissipative stress, or equivalently is $\sqrt{3}$ times the dissipative shear stress on the octahedral plane, and hence is, of course, independent of the load angle. The yield loci in true stress are hence given by Eqs. (20) or (23), where now $\bar{q} = q \equiv \|\mathrm{dev}\,\sigma\|$. They form a three parameter family of yield surfaces, determined by the values of μ, α and γ. In each case the critical state surface is now the Drucker–Prager cone, with a *circular* cross-section, given by (19). The cone angle is also the friction angle defined on the octahedral plane

$$\tan\phi_{\mathrm{oct}} \equiv \frac{t_{\mathrm{t,oct}}}{t_{\mathrm{n,oct}}} = \frac{\sqrt{3}\|\mathrm{dev}\,\sigma\|}{\mathrm{tr}\,\sigma} \equiv \frac{q}{\sqrt{3}p} = \mu \tag{28}$$

Since $\mathrm{dev}\,\chi = \mathrm{dev}\,\sigma$, it follows from (27) that the flow rule is associated in the deviatoric direction. However from (21) it is seen to possess volumetric associativity, *only* when $\alpha = \gamma = 1$, in which case the yield surfaces are ellipsoids. This is the classical three-dimensional generalization of modified Cam clay given by Roscoe and Burland (1968).

6. The Matsuoka–Nakai model

6.1. Properties of the spatially mobilized plane (SMP)

In this model failure is assumed to occur on the spatially mobilized plane (SMP), which is the plane in principal stress space whose normal is in the direction of the vector $\sigma^{-1/2}$. It was proposed by Matsuoka and Nakai (1974) and Matsuoka (1976) and discussed and extended in a number of subsequent papers. It is the mechanically based failure model which arguably, best fits experimental data. The normal and shear stress components of stress on the SMP, and their ratio are respectively:

$$t_{\mathrm{n}} = \frac{3}{(\mathrm{tr}\,\sigma^{-1})}, \quad t_{\mathrm{t}} = \frac{\sqrt{(\mathrm{tr}\,\sigma\,\mathrm{tr}\,\sigma^{-1} - 9)}}{\mathrm{tr}\,\sigma^{-1}}, \quad \text{and} \quad \frac{t_{\mathrm{t}}}{t_{\mathrm{n}}} = \sqrt{\left(\frac{1}{9}\mathrm{tr}\,\sigma\,\mathrm{tr}\,\sigma^{-1} - 1\right)} \tag{29}$$

Failure is assumed to occur when this ratio reaches a critical value, $\mu \equiv \tan\phi_{\mathrm{smp}}$ say, so that

$$\mathrm{tr}(\sigma)\mathrm{tr}(\sigma^{-1}) \equiv I_1 I_2 / I_3 = 9(1+\mu^2) = 9\sec^2\phi_{\mathrm{smp}} \tag{30}$$

where I_i $(i = 1, 2, 3)$ are the standard stress invariants. For present purposes it is more convenient to work with the "trace invariants", since these are much easier to differentiate, as discussed by Prevost (1987) and here in Appendix A. As is well known, this equation defines a cone in principal stress space, whose cross-section is close to the Coulomb model's irregular hexagon, but with rounded corners. In passing we note that (30) can also be written as:

$$-\tfrac{1}{3}\mathrm{tr}[(\mathrm{dev}\,\sigma)(\mathrm{dev}\,\sigma^{-1})] \equiv -\tfrac{1}{3}\mathrm{tr}[\sigma(\mathrm{dev}\,\sigma^{-1})] \equiv -\tfrac{1}{3}\mathrm{tr}[\sigma^{-1}(\mathrm{dev}\,\sigma)] = \mu^2 \tag{30'}$$

6.2. Generation of yield surfaces and flow rule

We now apply the above general procedure to develop the equations of a three parameter family of yield surfaces, all of which have the Matsuoka–Nakai cone as their critical state surface.

The dissipation function, which gives this cone as the *yield surface* in a *purely frictional* material was given by Houlsby (1981, 1993) and discussed further by Collins and Houlsby (1997) and Houlsby and Puzrin (2000). In the present notation, this function is

$$\Phi = \mu[(\mathrm{tr}\,\sigma)(\mathrm{tr}(\sigma\dot{e}^{\mathrm{p}^2})) - (\mathrm{tr}(\sigma\dot{e}^{\mathrm{p}}))^2]^{1/2} \tag{31}$$

It would not seem to have been appreciated however, that this function is proportional to the plastic shear strain rate on the *dual kinematic plane* (DKP), which is defined to be the plane whose normal is $\sigma^{1/2}$, in contrast to the SMP, whose normal is in the direction $\sigma^{-1/2}$. We will now demonstrate that the generalized Matsuoka–Nakai model, in which shear failure at the critical state occurs on the SMP, can be generated by consideration of the dissipation during shearing on the DKP. It is at first surprising that we do not consider shearing and failure on the same plane, as in the case of the Drucker–Prager model. The reason for this step will become apparent below.

We will assume, for the present, that the plastic strain rate and true stress tensors are co-axial. This is not generally true, even though the material is isotropic, as mentioned above, and discussed in more detail in Section 8. The validity of this assumption must hence be checked, by demonstrating the internal consistency of the resulting theory. The magnitudes of the plastic strain-rate vector, and its normal and shear components of the on this DKP can be written:

$$\dot{e}^{\mathrm{p}}_{\mathrm{dkp}} = \sqrt{\frac{\mathrm{tr}(\sigma\dot{e}^{\mathrm{p}^2})}{\mathrm{tr}\,\sigma}}, \quad \dot{e}^{\mathrm{p}}_{\mathrm{n,dkp}} = \frac{\mathrm{tr}(\sigma\dot{e}^{\mathrm{p}})}{\mathrm{tr}(\sigma)}, \quad \text{and} \quad \dot{e}^{\mathrm{p}}_{\gamma,\mathrm{dkp}} = \frac{\sqrt{\mathrm{tr}\,\sigma\,\mathrm{tr}(\sigma\dot{e}^{\mathrm{p}^2}) - [\mathrm{tr}(\sigma\dot{e}^{\mathrm{p}})]^2}}{\mathrm{tr}\,\sigma} \tag{32}$$

It is to be noted that the value of this shear strain rate is independent of the trace of the plastic strain rate and so can also be written:

$$\dot{e}^{\mathrm{p}}_{\gamma,\mathrm{dkp}} = \frac{\sqrt{\mathrm{tr}\,\sigma\,\mathrm{tr}(\sigma(\mathrm{dev}\,\dot{e}^{\mathrm{p}})^2) - [\mathrm{tr}(\sigma(\mathrm{dev}\,\dot{e}^{\mathrm{p}}))]^2}}{\mathrm{tr}\,\sigma} \tag{33}$$

By analogy with the definition of the plastic shear strain rate for the Drucker–Prager model we define the plastic shear strain rate in the dissipation function in (7) to be $\dot{\bar{e}}^{\mathrm{p}}_{\gamma} = \sqrt{3}\dot{e}^{\mathrm{p}}_{\gamma,\mathrm{dkp}}$, so that Φ is given by:

$$\Phi = \sqrt{(A\dot{e}^{\mathrm{p}}_{\mathrm{v}})^2 + (B\dot{\bar{e}}^{\mathrm{p}}_{\gamma})^2} = \sqrt{A^2\dot{e}^{\mathrm{p}^2}_{\mathrm{v}} + 3B^2\{(\mathrm{tr}\,\sigma)(\mathrm{tr}(\sigma\mathrm{dev}\,\dot{e}^{\mathrm{p}^2})) - (\mathrm{tr}(\sigma\mathrm{dev}\,\dot{e}^{\mathrm{p}}))^2\}/(\mathrm{tr}\,\sigma)^2} \tag{34}$$

and the corresponding dissipative pressure and dissipative deviatoric stress are found, using (6) to be:

$$\pi \equiv \tfrac{1}{3}\mathrm{tr}\,\chi = \partial\Phi/\partial\dot{e}^{\mathrm{p}}_{\mathrm{v}} = A^2\dot{e}^{\mathrm{p}}_{\mathrm{v}}/\Phi, \quad \text{as in Eq. (8) and}$$
$$\mathrm{dev}\,\chi = \partial\Phi/\partial(\mathrm{dev}\,\dot{e}^{\mathrm{p}}) = 3B^2\{(\mathrm{tr}\,\sigma)\sigma\mathrm{dev}\,\dot{e}^{\mathrm{p}} - \mathrm{tr}(\sigma\mathrm{dev}\,\dot{e}^{\mathrm{p}})\sigma\}/(\mathrm{tr}\,\sigma)^2\Phi \tag{35}$$

where we have used the basic results for differentiating traces of products of co-axial tensors given here in Appendix A. Note that this deviatoric, dissipative stress is, *by construction*, the work conjugate of the deviatoric strain rate, and *not* of the shear strain rate $\dot{\bar{e}}^{\mathrm{p}}_{\gamma}$, so that $\bar{\tau}$ is *not* the magnitude of the deviatoric dissipative stress in this model. Note too, that, as already remarked, *the dissipative stress is not co-axial* with the plastic strain-rate tensor.

Multiplying (35) by σ^{-1} and equating the deviatoric part of both sides of the equation, enables us to invert this equation and find an expression for the deviatoric plastic strain rate in terms of stress:

$$\mathrm{dev}\,\dot{e}^{\mathrm{p}} = \Phi\mathrm{tr}\,\sigma\mathrm{dev}(\sigma^{-1}\mathrm{dev}\,\chi)/3B^2 = -\Phi(\mathrm{tr}\,\sigma)^2\mathrm{dev}(\sigma^{-1})/9B^2 \tag{36}$$

The last equation follows from the fact that $\mathrm{dev}\,\chi = \mathrm{dev}\,\sigma$, since the shift stress is isotropic. The deviatoric part of the plastic strain rate is hence in the direction of the deviator of σ^{-1}, instead of the deviator of σ as in the Drucker–Prager model. However Eq. (36) verifies that the plastic strain-rate tensor *is co-axial with the true stress* tensor, so that the theory is internally consistent. By differentiating (30) with respect to true stress, the deviatoric part of the normal to the Matsuoka–Nakai cone is found to be in the direction of the deviator of σ^{-2}, so that flow rule for this model does not posses the deviatoric associativity property. This is to be expected, since the general theory of Collins and Houlsby (1997) predicts that the flow rule is only associated in the deviatoric plane, when the dissipation function depends on the true stress, only through the first invariant $\mathrm{tr}\,\sigma$. This is not true for this model as seen from the expression for Φ in (34). Matsuoka et al. (1999) also conclude that the flow rule must be non-associated, though their argument is rather different to the present one.

In order to find $\bar{q} = \bar{\tau}$, the true and dissipative shear stress invariants conjugate to $\dot{\bar{e}}^{\mathrm{p}}_{\gamma}$, which occur in the yield conditions and flow rules expressed in various forms in Eqs. (10), (11), (20), (21) and (23), we firstly square both sides of Eq. (35), multiply by σ^{-1}, and take the trace, giving:

$$\mathrm{tr}(\sigma^{-1}(\mathrm{dev}\,\chi)^2) = 9B^4\{(\mathrm{tr}\,\sigma)\mathrm{tr}(\sigma\mathrm{dev}\,\dot{e}^{\mathrm{p}^2}) - (\mathrm{tr}(\sigma\mathrm{dev}\,\dot{e}^{\mathrm{p}}))^2\}/(\mathrm{tr}\,\sigma)^3\Phi^2 \tag{37}$$

so that, using (33):

$$\bar{\tau} \equiv \frac{1}{3}\sqrt{\mathrm{tr}\,\sigma\,\mathrm{tr}(\sigma^{-1}(\mathrm{dev}\,\chi)^2)} = B^2\dot{\bar{e}}^{\mathrm{p}}_{\gamma}/\Phi \tag{38}$$

By comparison with (8) we deduce that the $\bar{\tau}$, as *defined* in (38), is the required conjugate dissipative shear stress. Finally expressing this result in terms of true stresses and simplifying, we find:

$$\bar{q} \equiv \frac{1}{3}\sqrt{\mathrm{tr}\,\sigma\,\mathrm{tr}(\sigma^{-1}(\mathrm{dev}\,\sigma)^2)} = p\sqrt{\left(\frac{1}{9}\mathrm{tr}\,\sigma\,\mathrm{tr}\,\sigma^{-1} - 1\right)} \tag{39}$$

Comparing (39) with (19), we recover (30), showing that the parameter μ is, indeed, the friction coefficient on the SMP. Eq. (20), for the three parameter, family of yield surfaces for this family of models, can hence be rewritten:

$$\frac{\left(p-\frac{1}{2}\gamma p_c\right)^2}{\left[(1-\gamma)p+\frac{1}{2}\gamma p_c\right]^2}+\frac{p^2\left[\frac{1}{9}\mathrm{tr}\,\sigma\,\mathrm{tr}\,\sigma^{-1}-1\right]}{\mu^2\left[(1-\alpha)p+\frac{1}{2}\alpha\gamma p_c\right]^2}=1 \tag{40}$$

Some representative cross-sections of these yield loci in the tri-axial plane are shown in Fig. 4. The four cases chosen are for the same values of the parameters α and γ as used in Fig. 3.

It is of interest to investigate the failure criterion, which would be generated if we were to use the plastic shear strain rate on the SMP instead of on the DKP in the dissipation function (34). The equations involving the volumetric strain rates and the pressure terms are of course unaltered. Those involving the shear strain rates and shear stresses in Eqs. (34)–(38) are altered simply by replacing σ by σ^{-1} everywhere, so that (38) is replaced by:

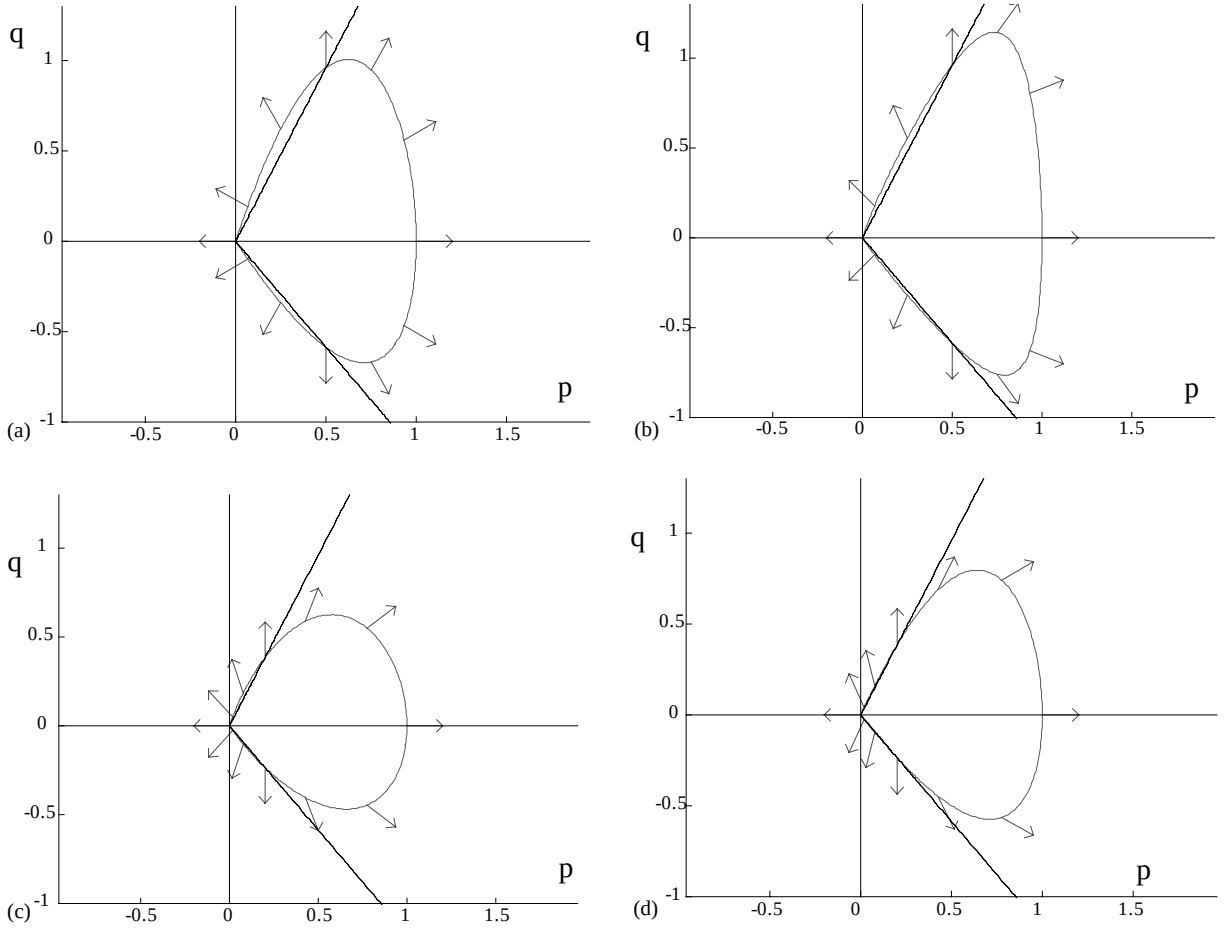

Fig. 4. The Matsoka–Nakai Model Yield loci in (p,q) space, for (a) $r=2$, $\alpha=1$ (Modified Cam clay); (b) $r=2$, $\alpha=0.5$; (c) $r=5$, $\alpha=0.5$; (d) $r=5$, $\alpha=0.2$, and $M=1$.

$$\bar{\tau} \equiv \frac{1}{3}\sqrt{\mathrm{tr}(\sigma^{-1})\mathrm{tr}(\sigma(\mathrm{dev}\,\chi)^2)} = B^2\dot{\bar{e}}^{\mathrm{p}}_{\gamma}/\Phi \tag{41}$$

so that now the true stress invariant is

$$\bar{q} \equiv \frac{1}{3}\sqrt{\mathrm{tr}(\sigma^{-1})\mathrm{tr}(\sigma(\mathrm{dev}\,\sigma)^2)} = \frac{1}{3}\sqrt{\mathrm{tr}(\sigma^{-1})\left[\mathrm{tr}\,\sigma^3 - \frac{2}{3}\mathrm{tr}\,\sigma^2\,\mathrm{tr}\,\sigma + \frac{1}{9}(\mathrm{tr}\,\sigma)^3\right]} \tag{42}$$

which is *not* the shear stress on the SMP. It is not the shear stress on the DKP either. This lack of symmetry arises because the relationship between the SMP and DKP is not perfectly symmetrical in the stress and plastic strain rate, since the normals to *both* planes are expressed in terms of the *stress tensor*. The possibility of considering dissipation induced on planes, whose normals are defined in terms of $\dot{e}^{\mathrm{p}}$ rather than σ, would be a worthwhile study, as it would produce models with deviatoric associativity, but this line of enquiry is not investigated further here.

Although this model is non-associative in the standard sense, it does possess a different kind of associativity. At the critical state the plastic volumetric strain rates are zero, by definition, so that $\dot{e}^{\mathrm{p}}$ can be replaced by $\mathrm{dev}\,\dot{e}^{\mathrm{p}}$ in (32) for the strain-rate components on the DKP. Hence using (36) we can obtain expressions for the normal and shear plastic strain rates on the DKP expressed entirely in terms of stress. The resulting expressions are:

$$\dot{e}_{\mathrm{n,dkp}} = (\Phi/3B^2)\mathrm{tr}\,\sigma\left(\frac{1}{9}\mathrm{tr}\,\sigma\,\mathrm{tr}\,\sigma^{-1} - 1\right) \quad \text{and} \quad \dot{e}_{\gamma,\mathrm{dkp}} = \pm(\Phi/3B^2)\mathrm{tr}\,\sigma\sqrt{\left(\frac{1}{9}\mathrm{tr}\,\sigma\,\mathrm{tr}\,\sigma^{-1} - 1\right)} \tag{43}$$

so that, at the critical state, the ratio of the normal to shear plastic, strain-rate components is just $\sqrt{\left(\frac{1}{9}\mathrm{tr}\,\sigma\,\mathrm{tr}\,\sigma^{-1} - 1\right)} = \mu \equiv \tan\phi_{\mathrm{smp}}$.

7. A note on work-conjugate planes

In order to understand the need to introduce the strain rates on the dual kinematic plane, rather than on the original spatially mobilized plane, consider the inner product of the traction vector on a plane whose normal is in the **n** direction in principal stress space with the plastic strain-rate vector on a plane whose normal is in the **m** direction in the same space. If the stress and plastic strain-rate tensors are assumed to be co-axial, these two vectors can be written:

$$\mathbf{t} = (\sigma_1 n_1, \sigma_2 n_2, \sigma_3 n_3)/\|\mathbf{n}\| \quad \text{and} \quad \dot{\mathbf{e}}^{\mathrm{p}} = (\dot{e}^{\mathrm{p}}_1 m_1, \dot{e}^{\mathrm{p}}_2 m_2, \dot{e}^{\mathrm{p}}_3 m_3)/\|\mathbf{m}\| \tag{44}$$

where $\|\ \|$ denotes the length of a vector. Their inner product is hence:

$$\mathbf{t}\cdot\dot{\mathbf{e}}^{\mathrm{p}} = \sum_{i=1}^{3}\sigma_i\dot{e}^{\mathrm{p}}_i m_i n_i/\|\mathbf{m}\|\|\mathbf{n}\| \tag{45}$$

If we now choose these planes to be *dual* in the sense that $m_i = n_i^{-1}$, for $i = 1, 2, 3$, so that $\mathbf{m}\cdot\mathbf{n} = 3$, then this expression can be rewritten in terms of the plastic work rate:

$$\widehat{W}^{\mathrm{p}} \equiv \mathrm{tr}(\sigma\dot{e}^{\mathrm{p}}) \equiv \sum_{i=1}^{3}\sigma_i\dot{e}^{\mathrm{p}}_i = (\mathbf{t}.\ \dot{\mathbf{e}}^{\mathrm{p}})(\|\mathbf{m}\|\|\mathbf{n}\|) = 3(\mathbf{t}.\ \dot{\mathbf{e}}^{\mathrm{p}})\sec\theta \tag{46}$$

where θ is the angle between the normals to the two planes. The two planes are hence also *work conjugate* in the sense that the product of the traction vector on one with the plastic strain-rate vector on the other gives the gives an expression for the plastic work rate.

This situation of having to consider deformations on one plane in order to develop failure conditions on another is typical of non-associated plasticity. It is a well known phenomena in classical plane strain theory

as discussed by Spencer (1964, 1982), De Josselin de Jong (1971), Collins (1990) and Anand (1983) for example, and has led to the development of "double shearing" models.

8. Comparison of thermo-mechanical and the extant potential procedures for formulating non-associated flow rules

8.1. The standard formulation

The standard procedure of writing the non-associated flow rule for an elastic/plastic material, is to introduce a plastic potential function $g(\sigma, e^{\mathrm{p}})$, and write the flow rule as:

$$\dot{e}^{\mathrm{p}} = \dot{\lambda}\frac{\partial g(\sigma, e^{\mathrm{p}})}{\partial \sigma} \tag{47}$$

Two objections have been leveled at this formulation:

(a) If one accepts that the plastic strain rate is a function of stress and plastic strain, in general there is no guarantee that this relation can be expressed in terms of the gradient of a potential function. Hunter (1976) has given a constructive proof of the existence of such a potential function, for an isotropic *incompressible* material, where the plastic strain rates depend on the second and third invariants of stress, but in general one cannot expect such a potential to exist, when the incompressibility condition is relaxed. In general plastic potentials can be guaranteed to exist in "two-dimensional" situations, where g depends on two components or two invariants of σ, but not in higher dimensional models. See Vardoulakis and Sulem (1995) for further discussion of this issue.

(b) The second objection relates to the coincidence of the principal axes of stress and plastic strain rate. This is a necessary consequence of (47) when the material is assumed isotropic. However there is ample experimental evidence demonstrating that this coincidence is not true when these principal axes rotate. This can, of course be explained in terms of the development of anisotropy. However the coincidence of these axes is no longer necessary, even for an isotropic material, when the potential depends on some other tensor in addition to stress, since the function g can depend on the joint invariants of σ and this second tensor. This has led Spencer (1964, 1982) and others to develop double shearing models in which the plastic strain rates depend on the *stress rates* as well as on the stress itself.

8.2. The thermo-mechanical (hyper-plastic) formulation

The new thermo-mechanical or hyper-plastic theory offers a different solution to these difficulties. If the yield function in dissipative space is denoted by $f(\sigma, e^{\mathrm{p}}; \chi)$, the yield function in true stress space is $\bar{f}(\sigma, e^{\mathrm{p}})$, where:

$$\bar{f}(\sigma, e^{\mathrm{p}}) \equiv f(\sigma, e^{\mathrm{p}}; \sigma - s(e^{\mathrm{p}})) \tag{48}$$

where we have used (6) to express the dissipative stress in terms of the true and shift stresses. As shown by Collins and Houlsby (1997) the plastic strain rates are then given by:

$$\dot{e}^{\mathrm{p}} = \dot{\lambda}\frac{\partial \bar{f}(\sigma, e^{\mathrm{p}})}{\partial \sigma} + \dot{\varepsilon}^{\mathrm{p}} \tag{49}$$

where the "additional plastic strain rate" $\dot{\varepsilon}^{\mathrm{p}}$ can be expressed either as:

$$\dot{\varepsilon}^{\mathrm{p}} = -\dot{\lambda}\frac{\partial f(\sigma, e^{\mathrm{p}}; \chi)}{\partial \sigma} \quad \text{or} \quad \dot{\varepsilon}^{\mathrm{p}} = \frac{\partial \Phi(\sigma, e^{\mathrm{p}}; \dot{e}^{\mathrm{p}})}{\partial \sigma} \tag{50}$$

Note that Desai and Siriwardane (1980) suggested that non-associativity could be modeled by adding a "correction function" to the standard normal flow rule equation. However the theory presented in Collins and Houlsby (1997) actually *proves* this result, and gives the explicit formula (50) for this correction term.

In the first form of Eq. (50), the dissipative stress χ is kept constant during the differentiation, and only put equal to $\sigma - s(e^{\mathrm{p}})$, *after* this differentiation has been performed. Hence in general one cannot expect to find a plastic potential function g, so that (49) can be rewritten in the form (47). However this is of no consequence, as the direction of the plastic strain-rate vector is readily evaluated by the transformation procedure illustrated above and in Collins and Kelly (2002) and Collins and Hilder (2002). There is no need to introduce a plastic potential in the thermo-mechanical formulation.

A possible resolution to the second issue, can be seen by studying the second form (50) for the additional plastic strain rate. The expression for the dissipation function Φ will, in general, involve the joint invariants of the stress and plastic strain-rate tensor, as already illustrated by (31) and (34), so that the principal axes of $\dot{\varepsilon}^{\mathrm{p}}$, will not, *necessarily*, coincide with the principal axes of either σ or $\dot{e}^{\mathrm{p}}$. However, in the case of the dissipation function for the Matsuoka–Nakai model (31), the additional plastic strain rate is:

$$\dot{\varepsilon}^{\mathrm{p}} \equiv \frac{\partial \Phi}{\partial \sigma} = \mu^2[\mathrm{tr}(\sigma \dot{e}^{\mathrm{p}^2})\delta + (\mathrm{tr}\,\sigma)\dot{e}^{\mathrm{p}^2} - 2\mathrm{tr}(\sigma \dot{e}^{\mathrm{p}})\dot{e}^{\mathrm{p}}]/\Phi \tag{51}$$

which is *co-axial* with the total plastic strain rate, so that from (49), σ and $\dot{e}^{\mathrm{p}}$ are also co-axial in this model. This justifies the co-axial assumption made above.

As an example of a non-co-axial model, consider the dissipation function:

$$\Phi = \mu\sqrt{\mathrm{tr}(\sigma \dot{e}^{\mathrm{p}} \sigma \dot{e}^{\mathrm{p}})} \tag{52}$$

The resulting dissipative stress and additional plastic strain rates are hence:

$$\chi = \frac{\partial \Phi}{\partial \dot{e}^{\mathrm{p}}} = (\mu^2/\Phi)(\sigma \dot{e}^{\mathrm{p}} \sigma) \quad \text{and} \quad \dot{\varepsilon}^{\mathrm{p}} = \frac{\partial \Phi}{\partial \sigma} = (\mu^2/\Phi)(\dot{e}^{\mathrm{p}} \sigma \dot{e}^{\mathrm{p}}) \tag{53}$$

From which we deduce that:

$$\dot{e}^{\mathrm{p}} = (\Phi/\mu^2)(\sigma^{-1}\chi\sigma^{-1}) \quad \text{and} \quad \sigma = (\Phi/\mu^2)(\dot{e}^{\mathrm{p}^{-1}} \dot{\varepsilon}^{\mathrm{p}} \dot{e}^{\mathrm{p}^{-1}}) \tag{54}$$

and

$$\mathrm{tr}(\chi\sigma^{-1}\chi\sigma^{-1}) = \mu^2 \quad \text{and} (\dot{e}^{\mathrm{p}^{-1}} \dot{\varepsilon}^{\mathrm{p}} \dot{e}^{\mathrm{p}^{-1}} \dot{\varepsilon}^{\mathrm{p}}) = \mu^2 \tag{55}$$

the first equations in (54) and (55) are the flow rule and yield condition in dissipative stress space. The dissipative stress is given by $\chi = \sigma - s(e^{\mathrm{p}})$, so that if the principal axes of plastic strain and plastic strain rate do not coincide, it follows from the first equation (54) that the principal axes of stress and plastic strain rate also do not coincide.

9. Discussion and conclusions

It has been one of the main objectives of this paper to further demonstrate that elastic/plastic models for geomaterials can be constructed in a systematic manner, by first constructing the yield condition and associated flow rule in dissipative stress space, and then mapping both into the true stress plane. Formally, this procedure is very similar to the transformation procedure adopted by Matsuoka et al. (1999) who used a geometric, mapping rule to transform the axisymmetric, (Drucker–Prager), three-dimensional versions of the original and modified Cam-clay models into models, whose critical surfaces are the Matsuoka–Nakai cones. The Drucker–Prager model is based on the use of the deviatoric stress invariant q, whereas the Matsuoka–Nakai model uses the stress invariant $\bar{q}$ as defined in (39). The mapping resulting from replacing

q by $\bar{q}$, is identical with that corresponding to transforming from dissipative to true stress space. This procedure can be generalized to generate other models, simply by redefining $\bar{q}$. It is argued, however, that the present approach has the advantage of being physically based, and the various procedural steps and "intermediate variables" can be given physical interpretations. It is also more general in that not only Cam clay, but any other family of (α, γ) tri-axial models developed by Collins and Kelly (2002) and Collins and Hilder (2002), can be generalized to three dimensions. It is also to be noted that the *original* Cam clay and its generalizations given by Matsuoka et al. (1999), are not acceptable in the themo-dynamic context, since they violate the second law of thermo-dynamics, by allowing plastic volume changes to occur without dissipating any energy, as discussed by Collins and Kelly (2002).

A rather similar geometrical mapping technique has been used by Gajo and Muir Wood (1999) to generate anisotropic, bounding surface models. These mappings can also be interpreted physically in terms of the mapping of rotating elliptical yield loci in dissipative stress space into the actual yield loci in true stress space, as described by Collins and Hilder (2002).

All of the models discussed in this paper, and the two earlier ones in this series are based upon the simple dissipation function in (7). The choice of this function is appropriate, if the plastic dissipation can be regarded as arising from volumetric compaction or expansion and from frictional shearing, corresponding to the two terms in the expression for Φ. It has been demonstrated that a large number of familiar as well as some new models can be generated from this simple function. The philosophy of the investigation has been simply to explore the consequences of assuming such a simple, but physically meaningful, dissipation model. An alternative approach has been adopted by Hashiguchi (2001), who proposed quite complicated forms for the dissipation function, and which have no direct physical interpretation, in order to generate elastic/plastic models with particular properties. Unfortunately, these dissipation functions do not satisfy the fundamental requirement of being non-negative over the entire range of the state variables, so that the conclusions of this paper, which question the worth of the thermo-mechanical procedure, are invalid.

A rather more valid criticism of the thermo-mechanical procedure is that it starts from assumptions of the forms of the two thermo-mechanical potentials (free energy and dissipation function), which cannot be determined directly in the laboratory. However instead, as argued above, we base the choice of these functions on our understanding of the fundamental mechanisms of energy storage and dissipation. In the end however, the validity of these basic assumptions, must be tested by comparing the predictions of the resulting theory with experimental observations. A particular observation, which at first sight is at odds with the concepts introduced by the thermo-mechanical theory, is that in shear deformations of granular materials all the plastic work would appear to be dissipated, and yet the thermo-mechanical models require at least some of the plastic work to be stored. This apparent paradox has been resolved by Collins and Muhunthan (2003), who, noting that the stored work is only required for *isotropic compaction processes*, demonstrated that the thermo-mechanical theory actually *predicts* that no plastic work is stored during those "quasi-steady" shear deformations which do not posses any normal compaction component, but the shift stresses are nevertheless non-zero.

Appendix A. Use of trace invariants

The standard set of fundamental stress invariants, used in continuum plasticity theory is I_1, I_2, I_3, defined as the coefficients in the characteristic equation for the principal stresses:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3\delta = 0 \tag{A.1}$$

However in problems involving the differentiation of these invariants it is preferable to work with a fundamental set of *trace* invariants, such as $\mathrm{tr}\,\sigma$, $\mathrm{tr}\,\sigma^2$, $\mathrm{tr}\,\sigma^3$, or $\mathrm{tr}\,\sigma^{-1}$, $\mathrm{tr}\,\sigma$, $\mathrm{tr}\,\sigma^2$. The properties of these invariants have been reviewed by Spencer (1971) and Prevost (1987). Those relevant to the present paper are:

$$\frac{\partial(\operatorname{tr}\sigma)}{\partial\sigma} = \delta, \quad \frac{\partial(\operatorname{tr}\sigma^n)}{\partial\sigma} = n\sigma^{n-1}(n \neq 1), \quad \frac{\partial(\operatorname{tr}\sigma\alpha)}{\partial\sigma} = \alpha, \quad \text{and} \quad \frac{\partial(\operatorname{tr}\sigma^2\alpha)}{\partial\sigma} = \sigma\alpha + \alpha\sigma \tag{A.2}$$

and that the value of the trace of a product of tensors is unchanged if the arrangement of the tensors undergoes a cyclic interchange, and, in the case of symmetric tensors, if the order is reversed.

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